Eur. Phys. J. B **1**, 245–253 (1998)

THE EUROPEAN PHYSICAL JOURNAL B

Nonlinear Schrödinger equation and superfluid hydrodynamics

C. Coste[a]

Laboratoire de Physique, ENS-Lyon[b], 46 allée d'Italie, 69364 Lyon Cedex 07, France

Received: 19 August 1997 / Revised: 16 October 1997 / Accepted: 6 November 1997

Abstract. We show that it is possible to generalize the Gross-Pitaevskii equation describing superfluidity in order to recover the two-fluid model, in the hydrodynamic limit, when the deviations from the equilibrium state are of long wavelength. When short distances are relevant, it is possible to keep trace of the purely quantum, non-hydrodynamic term of the Gross-Pitaevskii equation, so that the Hills-Roberts model, which describes the healing phenomenon, is finally obtained.

PACS. 67.40.Bz Phenomenology and two-fluid models

1 Introduction

Recently, there has been much interest in the investigation of superfluid properties based on the nonlinear Schrödinger equation (see below Eq. (2)), such as the excitation spectrum [1], sound scattering by quantum vortices [2], vortex nucleation [3] and liquid/vapour coexistence [4]. The nonlinear Schrödinger equation is a model of the *weakly* interacting Bose *gas* that has been proposed long times ago by Gross [5] and Pitaevskii [6]. In this model (hereafter G-P model), the temperature is set equal to zero, and the weakness of the interactions ensures that almost all helium atoms are Bose-condensed in the state of lowest energy. They all are in the same quantum state, so that a macroscopic wave-function appears very naturally; likewise, the superfluid velocity field is given by considering the probability flow of this wave function, and is proportional to the gradient of its phase.

Helium II is a dense fluid of strongly interacting bosons, so that the relevance of the G-P model to the description of superfluid helium is questionable, but this simplified model keeps the essential characteristics of superfluid dynamics. Moreover, there are now experimental realization of Bose condensed gas [7], which reinforces the relevance of the model. It is also possible, in the framework of the G-P model, to give a precise meaning to the concept of the *healing length*, the typical length scale on which the superfluid density increases from zero at the wall to its value in the bulk of the fluid. The requirement of a null superfluid density at a solid wall is necessary to exclude discontinuities in the superfluid mass flux, in agreement with some strong physical arguments (see [8,9] for a full discussion).

However, the Gross-Pitaevskii model is valid at zero temperature only, so that the normal component of the fluid, which behaves like an ordinary viscous fluid, is completely absent from the theory. The very peculiar properties of superfluid helium II are deeply related to the Bose condensation, which imply that a *macroscopic* part of the fluid is in the same quantum state. As shown by Feynmann [8], the symmetry properties of the ground state wave function, and the dense packing of the liquid both imply that at low temperature the low-energy excitations of the fluid should be phonons. This makes the link with the Landau [9] picture of the normal fluid as a gas of non-interacting quasiparticles, which are phonons at low temperature[1]. This is the microscopic basis for the two-fluid model of Landau [10] and Khalatnikov [11], who derived phenomenological equations for a mixture of normal fluid and superfluid, which behaves like an irrotational ideal fluid. Those equations are supposed to be valid at any temperature, possibly with the exception of the vicinity of the λ-point [12]. They do not describe the healing phenomenon, and they have been generalized in this sense by Hills and Roberts [13].

At zero temperature, the G-P coincide with the two-fluid model up to a term that contains derivatives of the superfluid density, and which is equal to the correction of Hills and Roberts. It is thus very tempting to generalize the equation of Gross and Pitaevskii, to include both the degrees of freedom describing the normal part of the fluid while keeping the non-hydrodynamic term responsible for healing near solid boundaries. The phenomenon of superfluidity is due to the macroscopic size of the number of particles in the *condensate*, which is the ground state of the fluid; the particles in the excited states of the gas form the *depletion*. In the G-P theory, there are only

[1] At higher temperature, another type of quasiparticles intervene, the rotons. It seems dangerous to extrapolate our work near the λ-point, but the rotons contribution to the thermodynamic functions is in principle easy to introduce in our calculations.

[a] e-mail: `coste@physique.ens-lyon.fr`
[b] URA CNRS 1325

weak interactions between the particles, and the depletion is completely neglected. A possible generalization of the G-P equation consists in taking into account the depletion at "order one", the main difficulty being to give a precise meaning to a development in power of the depletion. In previous attempts [14,15], the *global* gauge invariance of the G-P equation was used to introduce new degrees of freedom with the requirement of *local* gauge invariance, formally achieved by using the minimal coupling as in electrodynamic theory; the supplementary terms introduced in the equations were interpreted as properties of the depletion. In so doing, one does not obtain the two-fluid description of superfluidity; it is not surprising because the depletion *cannot* be identified with the normal component of the fluid, neither the condensate with its superfluid component. A more serious problem of those models is that they do not respect Galilean invariance.

Our aim is much more modest; we show that the minimal coupling technique, yet without any attempt to realize local gauge invariance, is sufficient to derive the two-fluid equations from the G-P model, in the hydrodynamic limit, and also to describe the healing phenomenon when shorter distances are relevant. We briefly review the G-P equation and the two-fluid model in Section 2. In Section 3, we derive the two-fluid model from the G-P equation. The Hills-Roberts theory is briefly reviewed in Section 4, where we show how to derive it from the G-P model. A summary and our conclusions are given in Section 5.

2 The Gross-Pitaevskii equation and the two-fluid model

In this section, we review briefly for completeness and further reference those two well-known theories of superfluidity.

The G-P equation [5,6] is obtained from a Hartree-Fock approach of the weakly interacting Bose gas. Assuming pair interaction $U(x-x')$ between the particles at positions x and x', one finds that the condensate wave function $\Psi(x,t)$ satisfies the equation:

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi - \mu_\infty\Psi + \Psi(x,t) \times \int |\Psi(x',t)|^2 U(x-x')\mathrm{d}^3x'. \quad (1)$$

This equation follows from the minimization of the energy of the fluid rather than its free energy, and in this respect it is valid only at absolute zero. If we suppose that the potential is of very short range and can be approximated by $U_0\delta(x-x')$, and use the fact that μ_∞, the chemical potential at infinity[2], taken at zero temperature, is equal to $(N_0/V_0)U_0$, where $N_0 = \int|\Psi|^2\mathrm{d}^3x$ is the number of particles in the condensate and V_0 the total volume occupied by the fluid, we find that Ψ satisfy the Nonlinear Schrödinger Equation (NLS):

$$i\partial_t\Psi = -\frac{1}{2}\nabla^2\Psi + |\Psi|^2\Psi - \Psi. \quad (2)$$

Here the mean density $\rho_0 \equiv N_0/V_0$ is taken as unity, the unit length is $\hbar/\sqrt{\rho_0 m U_0}$ and the unit time is $\hbar/(\rho_0 U_0)$. From now on, we will use dimensionless variables.

Setting $\Psi(x,t) \equiv \sqrt{\rho(x,t)}\mathrm{e}^{i\theta(x,t)}$, and separating the real and imaginary part in (2), we get

$$\partial_t\rho + \nabla\cdot\rho v = 0, \quad (3a)$$

$$\partial_t v + \nabla\frac{v^2}{2} + \frac{1}{\rho}\nabla\frac{\rho^2}{2} = -\nabla\left(\frac{(\nabla\rho)^2}{8\rho^2} - \frac{\nabla^2\rho}{4\rho}\right), \quad (3b)$$

where $v \equiv \nabla\theta$. Except for the high-derivative terms on the right-hand side of (3b), which can be discarded in the hydrodynamic regime, the system (3) is identical to the Euler equations for an irrotational ideal fluid with a pressure $p(\rho) \equiv \rho^2/2$. The pressure depend only on ρ because the fluid is taken at absolute zero.

On the other hand, if we neglect all dissipative processes, the equations of the two-fluid model of superfluidity are [10]:

$$\partial_t\rho + \nabla\cdot J = 0, \quad (4a)$$

$$\partial_t v_\mathrm{s} + \nabla\left(\frac{v_\mathrm{s}^2}{2} + \mu\right) = 0, \quad (4b)$$

$$\partial_t S + \nabla\cdot S v_\mathrm{n} = 0, \quad (4c)$$

$$\partial_t J^i + \partial_j \Pi_{ij} = 0, \quad (4d)$$

where the momentum density J and the stress tensor Π_{ij} are given by

$$J \equiv \rho_\mathrm{n} v_\mathrm{n} + \rho_{\mathrm{s}V_\mathrm{s}}, \quad (5a)$$

$$\Pi_{ij} \equiv \rho_\mathrm{n} v_\mathrm{n}^i v_\mathrm{n}^j + \rho_\mathrm{s} v_\mathrm{s}^i v_\mathrm{s}^j + p\delta_{ij}. \quad (5b)$$

Here ρ_n (resp. ρ_s) is the density of the normal (resp. superfluid) part of the fluid, v_n (resp. v_s) is the velocity of the normal (resp. superfluid) part of the fluid and $\rho \equiv \rho_\mathrm{n} + \rho_\mathrm{s}$ is the density of the whole fluid. The following thermodynamic functions are also introduced: μ is the chemical potential of the fluid, S its entropy per unit volume and p its local pressure. Those two sets of equations, (4, 5), imply the conservation of energy

$$\partial_t E + \nabla\cdot Q = 0, \quad (6a)$$

where E is the energy density

$$E = \frac{1}{2}\rho v_\mathrm{s}^2 + \rho_\mathrm{n} v_\mathrm{s}\cdot(v_\mathrm{n} - v_\mathrm{s}) + E_0, \quad (6b)$$

and Q the energy flux

$$Q = \left(\mu + \frac{1}{2}v_\mathrm{s}^2\right)J + TSv_\mathrm{n} + \rho_\mathrm{n} v_\mathrm{n}(v_\mathrm{n}^2 - v_\mathrm{n}\cdot v_\mathrm{s}). \quad (6c)$$

[2] In so doing, we implicitly assume that the fluid is at rest at infinity, otherwise μ_∞ should depend on the fluid velocity at infinity.

Here T is the temperature and E_0 the energy density in the reference frame where the velocity of the superfluid motion of the given fluid element is zero; the system of equations (4) must be supplemented by the thermodynamic identities

$$dE_0 = \mu d\rho + T dS + (v_n - v_s) \cdot d[\rho_n(v_n - v_s)], \quad (7a)$$

$$E_0 = -p + TS + \mu\rho + \rho_n(v_n - v_s)^2. \quad (7b)$$

At zero temperature, there is no normal fluid and the entropy is zero, so that (4c) is identically satisfied. A simplification occurs in (5) when one inserts $\rho_n = 0$, and (7) gives $dp = \rho d\mu$ in the zero temperature limit, so that (4b, 4d) are identical in this limit. The system (4) thus reduces to (4a, 4b) and, if we assume $\rho = \rho_s$ and $v_s = \nabla\theta$, is formally identical with the the G-P equation (3) in the hydrodynamic limit, that is when the high-derivative terms on the right-hand side of (3b) are discarded. It is natural to identify the irrotational velocity field v_s with $v = \nabla\theta$. On the contrary, the condensate density $|\Psi|^2$ must not be confused with the density of the superfluid part of the Bose gas, ρ_s. Indeed at absolute zero the whole fluid is superfluid, but if there are interactions between the particles they cannot be all in the condensate. However, the G-P equation is valid in the limit where the depletion represents a negligible fraction of the fluid. It is thus consistent to identify the condensate density with that of the superfluid in the G-P equation, but this approximation is legitimate for the weakly interacting Bose gas only, and not for real helium II in which the condensate represents only 10% of the whole fluid [16].

The subject of the next section is to generalize the G-P equation to finite temperature in order to recover the two-fluid model in the hydrodynamic limit.

3 Derivation of the two-fluid model

The two-fluid model is a phenomenological theory in the following sense: It is assumed that the relevant thermodynamic functions, ρ_n, ρ, μ, T, S and p, are *known* functions of a couple of independent thermodynamic variables like, *e.g.*, (p, T) or (ρ, S); this implies the knowledge of both the equation of state for the fluid, and the evolution of the normal fluid density with the thermodynamic state of the fluid. In a superfluid, which can sustain macroscopic mass flow in a state of thermodynamic equilibrium, the relative velocity w of the two fluids is a third independent thermodynamic variable; the dependence on w is given by requiring Galilean covariance at low relative velocity, whereas at higher velocity it is necessary to know the dependence of ρ_n on w. Otherwise, the superfluid velocity v_s is irrotational, so that it is fixed by one scalar function Φ such that $v_s = \nabla\Phi$. In that sense, the system (4) is a complete system of 6 equations for the 6 unknowns v_n, Φ and any two thermodynamic variables.

There is *no* phenomenology in the G-P equation, which leads to explicit expressions for the dependence of μ and p on ρ, the only remaining thermodynamic variable at zero temperature. Our aim is to generalize the theory in order to recover the two-fluid model, and we will proceed in a phenomenological way, assuming the relevant thermodynamic functions at finite temperature to be known. The G-P equation describes the evolution of the density ρ and the superfluid velocity potential θ, so that four new degrees of freedom are necessary if one wants to get the complete set of equations of the two-fluid model. It is clear that we must introduce the normal fluid velocity v_n, which means three suplementary degrees of freedom. As we just said, ρ is the natural thermodynamic variable of the G-P equation. It is thus appropriate to take as the other independent thermodynamic variable the entropy per unit volume S, thus completing the set of new degrees of freedom.

Equation (2) can be derived from the Lagrangian density

$$\mathcal{L}_0(\Psi, \Psi^*) = \frac{i}{2}\left[\Psi(\partial_t\Psi)^* - \Psi^*(\partial_t\Psi)\right] + \frac{|\nabla\Psi|^2}{2} + \frac{1}{2}|\Psi|^4 - |\Psi|^2, \quad (8)$$

which is invariant under global gauge transformations

$$\Psi \longrightarrow \Psi' = \Psi e^{i\alpha}, \qquad \Psi^* \longrightarrow \Psi'^* = \Psi^* e^{-i\alpha}, \quad (9)$$

where α is a real constant, and Ψ^* means the complex conjugate of Ψ. One (formal) way to include a scalar field ϕ and a vector field A is to require *local* gauge invariance of the Lagrangian density (8) under time and space dependent gauge transformations $\alpha(x, t)$; this is accomplished with the replacement of the derivatives by

$$\partial_t \longrightarrow \partial_t + i\phi, \qquad \nabla \longrightarrow \nabla - iA, \quad (10)$$

assuming that the fields are transformed under the gauge transformation like

$$\phi \longrightarrow \phi' = \phi - \partial_t\alpha, \qquad A \longrightarrow A' = A + \nabla\alpha. \quad (11)$$

Chela-Flores [15] considered only space dependent gauge transformations $\alpha(x)$, and thus only the vector field, which he identified with the velocity field of the normal part of the fluid. As we shall see, this is incorrect and further reflexion is necessary to understand the significance of the vector and scalar fields to be introduced.

First, the identification of the vector field A with the velocity field of the normal part of the fluid violates Galilean invariance of the model. As is well known, the two-fluid equations can be derived by requiring that they must be invariant if we change from the Galilean coordinate system K to another Galilean system K' moving relatively to K at constant velocity V. Thus the action of a Galilean boost

$$\begin{cases} x' = x + Vt \\ t' = t, \end{cases} \qquad \begin{cases} \nabla' = \nabla \\ \partial_{t'} = \partial_t - V \cdot \nabla, \end{cases} \quad (12)$$

leaves the system (4) invariant. On the other hand, equation (2) is also invariant under a Galilean boost if the wave function is transformed like

$$\Psi(x,t) = \Psi'(x',t')\exp[i(-Vx' + (V^2/2)t')], \quad (13)$$

so that the phase gradient actually behaves like a velocity: $\nabla'\theta' = \nabla\theta + V$. Thus any modification of (2) must respect Galilean invariance in order to be consistent. The vector and scalar fields should behave under changes of Galilean coordinate systems like, respectively, the vector and scalar potential of electromagnetism, or equivalently like the gradient and the time derivative (see (12))

$$\begin{cases} A' = A \\ \phi' = \phi - V \cdot A. \end{cases} \quad (14)$$

We thus see that A is invariant, and cannot be identified with a velocity. In the following, we will maintain Galilean invariance, and a convenient way to do that is to introduce the fields A and ϕ in the same manner as Chela-Flores did, but of course with a different physical interpretation. We will also see that *consideration of local gauge invariance* is neither possible nor necessary; we just need two fields that behave under Galilean boosts like in equation (14), and introduce them in the Lagrangian density (8) with the replacement rules (10) in order to preserve the invariance of NLS under Galilean transformations.

An interpretation of the vector field A, consistent with (14), and introducing the required new degrees of freedom in (8) is

$$A \equiv \chi(\nabla\theta - v_{\rm n}), \quad (15)$$

where $v_{\rm n}$ is a vector field corresponding to the velocity field of the normal part of the fluid, and χ is a Galilean invariant scalar field. The simplest approach consistent with the considerations of the beginning of this section is to assume that χ is a function of the density and the entropy (per unit volume) only. For the time being, we do not make any assumption concerning the scalar field ϕ, so that most generally we suppose $\phi = \phi(\rho, \theta, v_{\rm n}, S)$.

We thus introduce four new degrees of freedom, S and $v_{\rm n}$, and our first task is to find the relationship between χ and $\rho_{\rm n}$. When we insert in (8) the fields A and ϕ, according to the transformation law (10), we obtain the new Lagrangian density

$$\mathcal{L}_1(\Psi, \Psi^*, S, v_{\rm n}) = \mathcal{L}_0(\Psi, \Psi^*) + |\Psi|^2\phi + \frac{i}{2}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) \cdot A + \frac{1}{2}A^2|\Psi|^2. \quad (16)$$

Now, if we use the definition (15) of A, using $\nabla\theta = (\Psi^*\nabla\Psi - \Psi\nabla\Psi^*)/(2|\Psi|^2)$, the result is a complicated expression depending on Ψ, Ψ^*, and their gradients; it is much more convenient to use the variables ρ, θ, χ and $v_{\rm n}$, making the change of variables $\Psi = \sqrt{\rho}e^{i\theta}$ directly in $\mathcal{L}_1$

$$\mathcal{L}_1[\rho, \chi(\rho,S), \theta, v_{\rm n}] = \underbrace{\rho\partial_t\theta + \frac{\rho^2}{2} - \rho + \frac{\rho}{2}(\nabla\theta)^2 + \frac{(\nabla\rho)^2}{8\rho}}_{\equiv\mathcal{L}_0} + \frac{\rho}{2}(\chi^2 - 2\chi)(\nabla\theta)^2 + \rho\chi(1-\chi)v_{\rm n}\cdot\nabla\theta + \frac{\rho}{2}\chi^2 v_{\rm n}^2 + \rho\phi(\rho, S, \nabla\theta, v_{\rm n}). \quad (17)$$

The underbraced term is just the Lagrangian density $\mathcal{L}_0$ expressed in the variables (ρ, θ). The zero temperature limit is formally obtained when χ, ϕ and $v_{\rm n}$ all goes to zero; it is thus obvious that in this limit $\mathcal{L}_1$ is identical to $\mathcal{L}_0$. The high-derivative term $(\nabla\rho)^2/(8\rho)$ is negligible in the hydrodynamic limit, and irrelevant in this section, so that we will suppress it henceforward. We postpone a discussion including the effect of this non-hydrodynamic term to the next section.

From (17), we obtain the Lagrange equation for θ

$$\partial_t\rho + \nabla\cdot\left[\rho(1-\chi)^2\nabla\theta + \rho\chi(1-\chi)v_{\rm n} + \rho\frac{\partial\phi}{\partial\nabla\theta}\right] = 0, \quad (18)$$

which should clearly be associated with mass conservation in the two-fluid model. This is fulfiled assuming

$$\partial\phi/\partial\nabla\theta = \chi v_{\rm n}, \quad (19)$$

and defining

$$\rho_{\rm n} = \rho\chi(\rho,S)[2 - \chi(\rho,S)]$$

and

$$\rho_{\rm s} = \rho[1 - \chi(\rho,S)]^2, \quad (20)$$

so that (18) becomes

$$\partial_t\rho + \nabla\cdot\left[\rho(1-\chi)^2\nabla\theta + \rho\chi(2-\chi)v_{\rm n}\right] = 0, \quad (21)$$

which is now formally identical to equation (4a). In order to preserve the Galilean invariance of the theory, ϕ must evolve under a Galilean transform as required by (14); if we also require that it fulfils the condition (19), its only possible structure is

$$\begin{aligned} \phi &= \varphi(\rho,S) + \chi v_{\rm n}\cdot(\nabla\theta - v_{\rm n}) \\ &= \varphi + v_{\rm n}\cdot A, \end{aligned} \quad (22)$$

where φ is a Galilean invariant scalar function, which may depend on ρ and S.

It is now easy to see that the requirement of local gauge invariance becomes inapropriate. The total density ρ is unchanged by any gauge transform, and physically the normal and superfluid density cannot change, so that χ must be invariant too. Under a local gauge transform $\Psi \longrightarrow \Psi' = \Psi e^{i\alpha(x,t)}$, the superfluid velocity become $\nabla(\theta+\alpha)$; in order to ensure the behaviour of A required by local gauge invariance (11), with A given by (15), the normal fluid velocity should become $v_\mathrm{n} \longrightarrow v'_\mathrm{n} = v_\mathrm{n} + [(\chi - 1)/\chi]\nabla\alpha$. Whatever the value of χ, the two velocities do not behave in the same manner under a local gauge transform, so that the relative velocity is not invariant: Such a transformation of the velocities must consequently be rejected. An essential point in the derivation of the new Lagrangian density $\mathcal{L}_1$ from $\mathcal{L}_0$ is that the minimal coupling of the vector field A and the scalar field ϕ to ρ and θ, using (10), *does not ensure local gauge invariance* for $\mathcal{L}_1$, but only Galilean covariance if the fields behave like in (14). But this is the only property that is required by physical considerations, so that there is no problem in dropping local gauge invariance from the theory.

If we insert the expression (22) in the Lagrangian density (17), the Lagrange equation for ρ is seen to be

$$\partial_t\theta + \frac{(\nabla\theta)^2}{2} + \mu = 0. \tag{23}$$

Applying the gradient operator on this equation, we obtain

$$\partial_t\nabla\theta + \nabla\left(\frac{(\nabla\theta)^2}{2} + \mu\right) = 0, \tag{24}$$

which is identical to the equation for v_s, (4b), if we identify v_s and $\nabla\theta$; μ is thus to be identified with the chemical potential, and is given by

$$\begin{aligned}\mu = \varphi + \rho\frac{\partial\varphi}{\partial\rho} + \rho - 1 \\ - \underbrace{\left[\frac{1}{2}\chi(2-\chi) + \rho(1-\chi)\frac{\partial\chi}{\partial\rho}\right]}_{=(1/2)(\partial\rho_\mathrm{n}/\partial\rho)} w^2,\end{aligned} \tag{25}$$

where we used (20) for the identification of the underbraced term.

An interesting feature of equation (25) is that it exhibits explicit dependence of the chemical potential μ on w^2; w^2 is a Galilean invariant scalar, and it is a thermodynamic variable in a superfluid, because superfluids can support macroscopic mass flow without entropy flow, that is in a state of thermodynamic equilibrium. Equation (25), which here comes from the minimization of $\mathcal{L}_1$, could have been derived from thermodynamic identities also[3].

The remaining set of equations is readily obtained if we express the normal velocity v_n in terms of the Clebsch potentials, just as in Geurst's [17] variational formulation of the two-fluid model. The complete Lagrangian density of our model then reads

$$\begin{aligned}\mathcal{L}_2(\rho, S, \theta, v_\mathrm{n}, \alpha, \beta, \gamma) = \rho\partial_t\theta + \frac{\rho^2}{2} + \rho\varphi(\rho, S) \\ - \rho + \frac{\rho}{2}(1-\chi)^2(\nabla\theta)^2 + \rho\chi(2-\chi)v_\mathrm{n}\cdot\nabla\theta \\ + \frac{\rho}{2}\chi(\chi-2)v_\mathrm{n}^2 + \alpha\left[\partial_t S + \nabla\cdot(Sv_\mathrm{n})\right] \\ + \gamma\left[\partial_t(\beta S) + \nabla\cdot(\beta S v_\mathrm{n})\right],\end{aligned} \tag{26}$$

where it is recalled that $\chi = \chi(\rho, S)$. For convenience, and further comparison with the calculations of the next section, we give also the expression of $\mathcal{L}_2$ as a function of the densities, using (20):

$$\begin{aligned}\mathcal{L}_2(\rho, \rho_\mathrm{n}, \rho_\mathrm{s}, \theta, v_\mathrm{n}, \alpha, \beta, \gamma) = \rho\partial_t\theta + \frac{\rho^2}{2} + \rho\varphi \\ - \rho + \frac{\rho_\mathrm{s}}{2}(\nabla\theta)^2 + \rho_\mathrm{n}v_\mathrm{n}\cdot\nabla\theta - \frac{\rho_\mathrm{n}}{2}v_\mathrm{n}^2 \\ + \alpha\left[\partial_t S + \nabla\cdot(Sv_\mathrm{n})\right] + \gamma\left[\partial_t(\beta S) + \nabla\cdot(\beta S v_\mathrm{n})\right].\end{aligned} \tag{27}$$

Two of the Lagrange equations derived from (26, 21, 23), have already been written; the remaining set is:

$$(\delta\beta)\quad \partial_t\gamma + v_\mathrm{n}\cdot\nabla\gamma = 0, \tag{28a}$$

$$(\delta v_\mathrm{n})\quad \frac{\rho_\mathrm{n}}{S}w = -\left(\nabla\alpha + \beta\nabla\gamma\right), \tag{28b}$$

$$\begin{aligned}(\delta S)\quad \partial_t\alpha + \beta\partial_t\gamma + v_\mathrm{n}\cdot(\nabla\alpha + \beta\nabla\gamma) \\ + \rho(1-\chi)\frac{\partial\chi}{\partial S}w^2 - \rho\frac{\partial\varphi}{\partial S} = 0,\end{aligned} \tag{28c}$$

$$(\delta\alpha)\quad \partial_t S + \nabla\cdot Sv_\mathrm{n} = 0, \tag{28d}$$

$$(\delta\gamma)\quad \partial_t\beta + v_\mathrm{n}\cdot\nabla\beta = 0. \tag{28e}$$

Two of the Clebsch potentials, α and γ, appear as Lagrange multipliers. The constraint related to α gives the conservation of entropy, (28d), and the constraint related to γ the conservation of *normal fluid* vorticity[4], (28a, 28e). However, we already know that, by definition, the superflow carries no entropy, hence that the normal fluid only is responsible for the entropy transport, as (28d) shows; the G-P equation also prescribes that the normal fluid vorticity must move with the normal fluid owing to the irrotational nature of the superfluid, in agreement with (28a, 28e). Thus the two constraints needed to introduce the Clebsch potentials actually represent no supplementary assumptions.

Equation (28b), which gives the relative velocity w in terms of the Clebsch potentials α, β and γ, has been obtained in the same form by Geurst [17]. This is not very surprising for two reasons. The first one is, of course, that we use the same parametrization for the normal fluid velocity. The second traces back to the fact that his theory is a generalization of the Lagrangian formulation of the irrotational ideal fluid hydrodynamics in Eulerian representation; on the other hand, NLS is a good parametrization of the Euler equation (discarding the terms with the highest derivatives, see (2, 3)), so that the Lagrangian density

[3] Equation (25) seems to differ from the corresponding one usually given in textbooks on superfluid hydrodynamics (see *e.g.*, [10]); the reader is reminded that our thermodynamic variables are not the usual ones, *i.e.*, T, p and w^2 but ρ, S and w^2.

[4] Not to be confused with the so-called quantum vortices, which may exist if the phase of the wavefunction is multivalued.

(8) is also a variational formulation of ideal fluid hydrodynamics. However, the conservation of mass is introduced by Geurst as a constraint equation, with the help of a Lagrange multiplier that is identified with the potential of the superfluid velocity; in the G-P theory, the mass conservation and the identification of the superfluid velocity potential are both direct consequences of the model, a feature that is preserved in our generalization.

The system (28) is complete and thus sufficient to derive the two-fluid equations; it is nevertheless quicker to obtain the conservation of momentum and energy, and thus the expression of the relevant thermodynamic functions, from the stress energy tensor

$$T_{\sigma\tau} = \frac{\partial \mathcal{L}_2}{\partial(\partial_\tau \eta_\lambda)}\, \partial_\sigma \eta_\lambda - \mathcal{L}_2 \delta_{\sigma\tau}, \tag{29}$$

where $\eta = (\rho, \theta, v_{\mathrm{n}}^i, S, \alpha, \beta, \gamma)$ is the set of fields in $\mathcal{L}_2$, indexed by λ in (29), and $\partial_\sigma = (\partial_0, \partial_i) = (\partial_t, \nabla_i)$ (not a 4-vector!). The components of the tensor $T_{\sigma\tau}$ satisfy the conservation laws

$$\partial_t(T_{\sigma 0}) + \nabla_i(T_{\sigma i}) = 0, \tag{30}$$

where σ runs from 0 to 4.

A straightforward calculation gives the energy density

$$\begin{aligned} -T_{00} &= E \\ &= \frac{1}{2}\rho v_{\mathrm{s}}^2 + \rho_{\mathrm{n}} v_{\mathrm{s}} \cdot w + \rho_{\mathrm{n}} w^2 + \mu\rho - p + TS + \nabla \cdot C, \end{aligned} \tag{31}$$

where C is the vector $(\alpha + \beta\gamma)\, S v_{\mathrm{n}}$. The divergence term $\nabla \cdot C$ and a corresponding term in $-T_{0i}$, $-\partial_t C_i$, cancel out each other when put in the energy conservation law given by (30), so that they just reflect the degree of arbitrariness in the definition of $T_{\sigma\tau}$. In (31) we also defined

$$-p + TS \equiv -\frac{\rho^2}{2} - \rho^2 \frac{\partial \varphi}{\partial \rho} + \rho^2 (1-\chi) \left(\frac{\partial \chi}{\partial \rho}\right) w^2. \tag{32}$$

With this definition, considering (7b, 6b), we see that $-T_{00}$ (apart from the spurious divergence term) is indeed identical with the energy density E. The energy current density is

$$\begin{aligned} -T_{0i} &= Q_i \\ &= \left(\mu + \frac{1}{2} v_{\mathrm{s}}^2\right) J_i + TS v_{\mathrm{n}}^i + \rho_{\mathrm{n}} v_{\mathrm{n}}^i (v_{\mathrm{n}}^2 - v_{\mathrm{n}} \cdot v_{\mathrm{s}}) - \partial_t C_i, \end{aligned} \tag{33}$$

if we define

$$T = \rho \frac{\partial \varphi}{\partial S} - \rho(1-\chi) \frac{\partial \chi}{\partial S} w^2. \tag{34}$$

The dependence of this thermodynamic function on w is the same as that of the temperature, given by general thermodynamic requirements, which proves the consistency of this definition. When we insert (31, 33) in (30), setting $\sigma = 0$, we recover the conservation of energy for the two-fluid model (6a, b, c).

The momentum density is

$$\begin{aligned} T_{i0} &= J_i \\ &= \rho_{\mathrm{s}} v_{\mathrm{s}}^i + \rho_{\mathrm{n}} v_{\mathrm{n}}^i + \nabla_i D, \end{aligned} \tag{35}$$

where D is the scalar $(\alpha + \beta\gamma)\, S$. When the conservation of momentum is expressed using (30), the gradient term and a corresponding term in T_{ij} cancel out each other; indeed T_{ij} reads

$$\begin{aligned} T_{ij} &= \Pi_{ij} \\ &= \rho_{\mathrm{n}} v_{\mathrm{n}}^i v_{\mathrm{n}}^j + \rho_{\mathrm{s}} v_{\mathrm{s}}^i v_{\mathrm{s}}^j + (p - \partial_t D)\delta_{ij}, \end{aligned} \tag{36}$$

where we define

$$\begin{aligned} p = {} & \frac{\rho^2}{2} + \rho^2 \frac{\partial \varphi}{\partial \rho} + \rho S \frac{\partial \varphi}{\partial S} \\ & - \rho(1-\chi) \left(S \frac{\partial \chi}{\partial S} + \rho \frac{\partial \chi}{\partial \rho}\right) w^2. \end{aligned} \tag{37}$$

This definition is consistent with equations (32, 34), and with the dependence of the pressure on w as required by general thermodynamic identities. Thus the thermodynamic functions that we introduced, with μ identified with the chemical potential, T with the temperature and p with the pressure, are fully consistent with the physical requirements of thermodynamics concerning their respective dependence on w^2. It reflects the fact that the Lagangian density (26) is fully covariant with respect to Galilean transforms. Inserting (35, 36) in (30), one recovers the conservation of momentum of the two-fluid model, (4d, 5). The set of equations (21, 24, 31, 33, 35, 36) definitions of thermodynamic quantities (25, 34, 37), are thus the equations of the two-fluid model, with the normal and superfluid density given by (20). As we said at the beginning of this section, the functions $\chi(\rho, S)$ and $\varphi(\rho, S)$ are not specified in our theory; they have to be deduced from a microscopic theory, or from experimental data.

4 Derivation of the Hills-Roberts model

The irrotational nature of the superfluid velocity leads to a paradox, most clearly stated in the review by Ginzburg and Sobaynin [18]; let us recall briefly their argument. As in any perfect fluid, the component of v_{s} parallel to a wall need not vanishes; on the other hand, helium atoms stick to the wall, so that the flux of superfluid[5] do vanish: $(\rho_{\mathrm{s}} v_{\mathrm{s}})|_{\mathrm{wall}} = 0$. So it seems that a discontinuity in v_{s} may be possible; but this should lead to the appearance of a measurable surface energy at $v_{\mathrm{s}} \neq 0$, which is not observed in experiments. A possible way out this contradiction is to assume that $\rho_{\mathrm{s}}|_{\mathrm{wall}} = 0$. The NHT of the Gross-Pitaevskii model allows the fulfilement of this boundary condition. In the context of the two-fluid model, terms depending on the gradients of the superfluid density have to be added

[5] The flux of normal fluid, $(\rho_{\mathrm{n}} v_{\mathrm{n}})|_{\mathrm{wall}}$, vanishes because $v_{\mathrm{n}}|_{\mathrm{wall}} = 0$ as for any other viscous fluid.

to the equations. This is done in the Hills-Roberts model [13]. In this model, equation (4b) is replaced by

$$\partial_t v_{\rm s} + \nabla\left(\frac{v_{\rm s}^2}{2} + \mu\right) = \\ = \nabla\left(\eta(\rho_{\rm s})\nabla^2\rho_{\rm s} + \frac{1}{2}\frac{{\rm d}\eta}{{\rm d}\rho_{\rm s}}(\nabla\rho_{\rm s})^2\right), \quad (38)$$

where $\eta(\rho_{\rm s})$ is an unspecified function; the stress tensor have to be modified also, and reads now

$$\Pi_{ij} \equiv \rho_{\rm n} v_{\rm n}^i v_{\rm n}^j + \rho_{\rm s} v_{\rm s}^i v_{\rm s}^j + \eta\nabla_i\rho_{\rm s}\nabla_j\rho_{\rm s} \\ + \left(p - \eta\rho_{\rm s}\nabla^2\rho_{\rm s} - \frac{1}{2}\frac{{\rm d}(\rho_{\rm s}\eta)}{{\rm d}\rho_{\rm s}}(\nabla\rho_{\rm s})^2\right)\delta_{ij}, \quad (39)$$

in contrast with its last expression (5b). The new terms, involving the spatial derivatives of $\rho_{\rm s}$, are responsible for the existence of a *healing length* near a wall: this is the typical distance over which the superfluid density evolves from 0 to its value in the bulk of the fluid. The two-fluid equations cannot account for this phenomenon, which is essentially of non-hydrodynamical nature.

In the zero temperature limit, the Hills-Roberts and G-P model are strictly identical, with $\eta(\rho_{\rm s}) = 1/(4\rho_{\rm s})$. To generalize the G-P model, and obtain the Hills-Roberts equations, we proceed with the same spirit as before, using the minimal coupling technique, and imposing Galilean invariance at all stages of the calculations. As we will see, although the details of the derivation seem quite different, the final result is very similar to what we got in the last section.

The natural thermodynamic variables, in the Hills-Roberts equations, are the superfluid density $\rho_{\rm s}$ and the entropy per unit volume S. We thus make a different change of variables, setting $\Psi = \sqrt{\rho_{\rm s}}{\rm e}^{i\theta}$, and get the following expression for the NLS Lagrangian density:

$$\widetilde{\mathcal{L}}_0(\rho_{\rm s},\theta) = \rho_{\rm s}\partial_t\theta + \frac{\rho_{\rm s}^2}{2} - \rho_{\rm s} + \frac{\rho_{\rm s}}{2}(\nabla\theta)^2 + \frac{(\nabla\rho_{\rm s})^2}{8\rho_{\rm s}}. \quad (40)$$

For the sake of clarity, the quantities similar to the ones of the previous section, but with a different interpretation, will be affected hereafter with a *tilde*: $\tilde{}$. We now introduce the vector and scalar potentials as follow:

$$\widetilde{A} = \widetilde{\chi}(\nabla\theta - v_{\rm n}), \\ \widetilde{\phi} = \chi_1\partial_t\theta + \chi_2(\nabla\theta)^2 + \chi_3\nabla\theta\cdot v_{\rm n} + \chi_4 v_{\rm n}^2 + \widetilde{\varphi}, \quad (41)$$

where all the unknown functions, $\widetilde{\chi}$, $\widetilde{\varphi}$ and the four χ_i are supposed to depend on the thermodynamic variables $\rho_{\rm s}$ and S. The structure of $\widetilde{\phi}$ is the most general one, quadratic in the velocities.

The next step is to insert the expressions (41) in the original Lagrangian with the minimal coupling assumption, like in (16). The variational equation for θ reads

$$\partial_t\rho_{\rm s} + \partial_t\left(\rho_{\rm s}\frac{\partial\widetilde{\phi}}{\partial(\partial_t\theta)}\right) \\ + \nabla\cdot\left[\rho_{\rm s}(1-\widetilde{\chi})^2\nabla\theta + \rho_{\rm s}\widetilde{\chi}(1-\widetilde{\chi})v_{\rm n} + \rho_{\rm s}\frac{\partial\widetilde{\phi}}{\partial(\nabla\theta)}\right] = 0; \quad (42)$$

it must be identified with the equation of mass conservation (4a), which gives the following set of conditions

$$\rho_{\rm n} \equiv \rho_{\rm s}\chi_1(\rho_{\rm s}, S), \quad (43a)$$
$$(1-\widetilde{\chi})^2 + 2\chi_2 = 1, \quad (43b)$$
$$\widetilde{\chi}(1-\widetilde{\chi}) + \chi_3 = \chi_1. \quad (43c)$$

Another set of conditions comes from the requirement of Galilean invariance of the theory, which means that $\widetilde{A}$ and $\widetilde{\phi}$ must obey the transformation law (14). Equating the coefficients of the terms in $\nabla\theta V$, $V v_{\rm n}$ and V^2, we obtain another set of conditions,

$$-\chi_1 + 2\chi_2 + \chi_3 = -\widetilde{\chi}, \quad (44a)$$
$$\chi_3 + 2\chi_4 = \widetilde{\chi}, \quad (44b)$$
$$\chi_2 + \chi_3 + \chi_4 = \frac{1}{2}\chi_1. \quad (44c)$$

The last equation, (44c), is not independent, being the sum of the two others. Using (43b, 43c), we get

$$-\chi_1 + 2\chi_2 + \chi_3 = \widetilde{\chi}, \quad (45)$$

which together with (44a) gives

$$\widetilde{\chi} = 0 \quad \Longrightarrow \quad \widetilde{A} = 0, \quad (46)$$

so that there is no vector potential anymore. This difference with the derivation of the last section is purely formal, in a sense, as we will see below. Using (46), we finally deduce

$$\chi_2 = 0, \\ \chi_3 = \chi_1, \\ \chi_4 = -\frac{1}{2}\chi_1. \quad (47)$$

As in the previous case, the model is completely described by the scalar functions $\chi_1(\rho_{\rm s}, S)$ and $\widetilde{\varphi}(\rho_{\rm s}, S)$, and the vector function $v_{\rm n}$.

With those results, we may calculate the variational equation for $\rho_{\rm s}$. It reads

$$\partial_t\nabla\theta + \nabla\left(\frac{(\nabla\theta)^2}{2} + \widetilde{\mu}\right) = \\ = \nabla\cdot\left[\left(\frac{\partial\rho_{\rm s}}{\partial\rho}\right)_S\left(\frac{\nabla^2\rho_{\rm s}}{4\rho_{\rm s}} - \frac{(\nabla\rho_{\rm s})^2}{8\rho_{\rm s}^2}\right)\right]. \quad (48)$$

At very low temperature, the derivative $(\partial\rho_{\rm s}/\partial\rho)_S$ is almost equal to 1, so that with this equation we recover the Hills-Roberts model (38), with $\eta(\rho_\mathrm{s}) = 1/4\rho_\mathrm{s}$. The new chemical potential, $\widetilde{\mu}$, is given by

$$\widetilde{\mu} = \left(\frac{\partial \rho_\mathrm{s}}{\partial \rho}\right)_S \left(\rho_\mathrm{s} - 1 + \frac{\partial(\rho_\mathrm{s}\widetilde{\varphi})}{\partial \rho_\mathrm{s}}\right) - \frac{1}{2}\frac{\partial \rho_\mathrm{n}}{\partial \rho} w^2, \qquad (49)$$

and it exhibits the correct explicit dependence on w^2 (see (25)).

In order to get the other equations of the model, we use the Clebsch potentials as before. The complete Lagrangian density, using the expression of $\widetilde{\mathcal{L}}_0$ given in (40), the minimal coupling with the potentials given by (41), and equations (43a, 47), now reads

$$\begin{aligned}\widetilde{\mathcal{L}}_2(\rho, \rho_\mathrm{n}, \rho_\mathrm{s}, \theta, v_\mathrm{n}, \alpha, \beta, \gamma) &= \rho \partial_t \theta + \frac{\rho_\mathrm{s}^2}{2} \\ &+ \rho_\mathrm{s}\widetilde{\varphi} - \rho_\mathrm{s} + \frac{\rho_\mathrm{s}}{2}(\nabla\theta)^2 + \rho_\mathrm{n} v_\mathrm{n}\cdot\nabla\theta - \frac{\rho_\mathrm{n}}{2}v_\mathrm{n}^2 \\ &+ \alpha\left[\partial_t S + \nabla\cdot(S v_\mathrm{n})\right] + \gamma\left[\partial_t(\beta S)\right. \\ &\left. + \nabla\cdot(\beta S v_\mathrm{n})\right] + \frac{\nabla^2 \rho_\mathrm{s}}{8\rho_\mathrm{s}}\cdot \end{aligned} \qquad (50)$$

Apart from the very last term accounting for the healing phenomenon, this expression is extremely similar to the corresponding one in the last section, (27). In this section, there is no vector potential, as shown by (46), so that the calculations may seem very different from the previous ones; but the final result, (50), is essentially the same.

Consequently, only one Lagrange equation is modified. The variationnal equation for the entropy density, previously given by (28c), now reads

$$\begin{aligned}(\delta S) \quad & \partial_t\alpha + \beta\partial_t\gamma + v_\mathrm{n}\cdot(\nabla\alpha + \beta\nabla\gamma) \\ & - \frac{\partial \rho_\mathrm{n}}{\partial S}\left(\partial_t\theta + v_\mathrm{n}\cdot\nabla\theta - \frac{v_\mathrm{n}^2}{2}\right) - \rho_\mathrm{s}\frac{\partial\widetilde{\varphi}}{\partial S} = 0. \end{aligned} \qquad (51)$$

The remaining calculations proceed just as in the previous section. The new expression for the stress tensor $\widetilde{\Pi}_{ij}$ reads

$$\begin{aligned}\widetilde{\Pi}_{ij} &= \rho_\mathrm{n} v_\mathrm{n}^i v_\mathrm{n}^j + \rho_\mathrm{s} v_\mathrm{s}^i v_\mathrm{s}^j + \frac{1}{4\rho_\mathrm{s}}\nabla_i\rho_\mathrm{s}\nabla_j\rho_\mathrm{s} \\ &+ \left[\widetilde{p} - \rho\frac{\partial\rho_\mathrm{s}}{\partial\rho}\left(\frac{\nabla^2\rho_\mathrm{s}}{4\rho_\mathrm{s}} - \frac{(\nabla\rho_\mathrm{s})^2}{4\rho_\mathrm{s}^2}\right) - \frac{(\nabla\rho_\mathrm{s})^2}{8\rho_\mathrm{s}}\right]\delta_{ij}, \end{aligned} \qquad (52)$$

where the pressure $\widetilde{p}$ is given by

$$\begin{aligned}\widetilde{p} &= \widetilde{T}S + \rho^2\frac{\partial}{\partial\rho}\left(\frac{\rho_\mathrm{s}^2/2 - \rho_\mathrm{s} + \rho_\mathrm{s}\widetilde{\varphi}}{\rho}\right) \\ &- \frac{\rho^2}{2}\frac{\partial}{\partial\rho}\left(\frac{\rho_\mathrm{n}}{\rho}\right)w^2, \end{aligned} \qquad (53)$$

and the temperature $\widetilde{T}$ by

$$\widetilde{T} = \left(\frac{\partial\rho_\mathrm{s}}{\partial S}\right)_\rho \left(\rho_\mathrm{s} - 1 + \frac{\partial(\rho_\mathrm{s}\widetilde{\varphi})}{\partial\rho_\mathrm{s}} - \frac{w^2}{2}\right) + \frac{\partial(\rho_\mathrm{s}\widetilde{\varphi})}{\partial S}\cdot \qquad (54)$$

The term depending on the superfluid density gradient in (52) is almost the same as in the Hills-Roberts model (39). We insist that our calculations are supposed to be valid at very low temperature, where $\rho_\mathrm{s} \approx \rho \approx \rho(\partial\rho_\mathrm{s}/\partial\rho)$. We have thus obtained from the G-P equation, in a consistent way, the equations of the Hills-Roberts model of superfluidity.

5 Summary and conclusions

The Gross-Pitaevski model describes the superfluidity of the weakly interacting Bose gas. More precisely, it gives the behavior of the wave function of the condensate, which is the part of the fluid that undergoes Bose condensation, and neglects the other part of the fluid, the depletion. Although approximate, this theory involves no phenomenology, but is valid at zero temperature only. It accounts for both superfluid behavior in the bulk of the fluid, and healing when the fluid is in contact with a solid wall.

At low, but non zero temperature, helium II is in a superfluid phase. The behavior of this superfluid liquid, with rather strong interatomic interactions, is very much the same as the one of a mixture of a normal viscous fluid and an irrotational ideal fluid. This is the basis of the phenomenological two-fluid model of superfluidity. In this model, gradients of the superfluid density do not intervene, so that the healing phenomenon is not taken into account. This task is undertaken by the Hills-Roberts theory, which add to the two-fluid equations terms that depend on the superfluid density gradient. At zero temperature, the Hills-Roberts and G-P models are identical, and their hydrodynamic limit is the two-fluid model.

In this paper, we show that it is possible to add the degrees of freedom that correspond to the normal fluid, starting from the Lagrangian of the G-P model. We proceed in a systematic way, ensuring Galilean covariance at all stages of the calculation. In the hydrodynamic limit, we get the equations of the two-fluid model, with consistent definitions for all the relevant thermodynamic functions of the fluid. When the derivatives of the superfluid density are no more negligible, a similar method lead to the equations of the Hills-Roberts theory. The details of the calculations are somewhat different, but the final result is essentially the same, apart from the new terms depending on the superfluid density gradient. Our hope is that such calculations may be useful for a more systematic derivation of the two-fluid or Hills-Roberts theory, in which the G-P equation should be the first order of a perturbative approach in power of the depletion.

I am grateful to Pr. S. Fauve for constant interest and encouragement during the completion of this work.

References

1. Y. Pomeau, S. Rica, Phys. Rev. Lett. **69**, 1644 (1992).
2. C. Nore, M.E. Brachet, E. Cerda, E. Tirapegui, Phys. Rev. Lett. **72**, 2593 (1994).

3. C. Josserand, Y. Pomeau, S. Rica, Phys. Rev. Lett. **75**, 3150 (1995).
4. C. Josserand, S. Rica, Phys. Rev. Lett. **78**, 1215 (1997).
5. E.P. Gross, Nuovo Cimento **20**, 454 (1961).
6. L.P. Pitaevskii, Sov. Phys. JETP **13**, 451 (1961).
7. C.C. Bradley, C.A. Sackett, J.J. Tollet, R.G. Hulet, Phys. Rev. Lett. **75**, 1687 (1995); W. Petrich, M.H. Anderson, J.R. Ensher, E.A. Cornell, Phys. Rev. Lett. **75**, 3352 (1995); K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, W. Ketterle, Phys. Rev. Lett. **75**, 3969 (1995).
8. R.P. Feynmann, *Progress in low temperature physics (1)*, edited by C.J. Gorter (1955); See also R.P. Feynmann, *Statistical Mechanics* (Benjamin-Cummings, New York, 1982)
9. L.D. Landau, J. Phys. France **5**, 71 (1941); J. Phys. France **11**, 91 (1947). See the reprint section at the end of ref. [11].
10. L.D. Landau, E.M. Lifshitz, *Fluid Mechanics* (Pergamon, Oxford, 1987).
11. I.M. Khalatnikov, *An introduction to the theory of superfluidity* (Benjamin, New York, 1965).
12. V.L. Ginzburg, L.P. Pitaevskii, Sov. Phys. JETP **34**, 858 (1958).
13. R.N. Hills, P.H. Roberts, Int. J. Engng. Sci. **15**, 305 (1977); J. Low Temp. Phys. **30**(5/6), 709 (1978); J. Phys. C **11**, 4485 (1978).
14. F.W. Cummings, J.S. Herold, J. Sarfatt, Physica **50**, 15 (1970).
15. J. Chela-Flores, J. Low Temp. Phys. **21**, 307 (1975).
16. O. Penrose, L. Onsager, Phys. Rev. **104**, 576 (1956).
17. J.A. Geurst, Phys. Rev. B **22** 3207 (1980).
18. V.L. Ginzburg, A.A. Sobaynin, Sov. Phys. Usp. **19**(10), 773 (1976).